Math 2A – Chapter 10 Sampling of Problems to study for the Test – Fall '07

1. Find a parameterization for the parabola formed by the intersection of the plane $z = x + 3y$ with the cone $z = 2\sqrt{x^2 + y^2}$.
2. Find a parameterization of the line segment from $\vec{r}\ \ 1$ to $\vec{r}\ \ 2$ where $r\ \ t\ = \langle t, t^2, t^4 \rangle$.
3. Parameterize the curve at the intersection of the ellipsoid $x^2 + 9y^2 + 4z^2 = 1$ with the elliptical paraboloid $x = 9y^2 + 4z^2$.
4. Find parametric equations for the tangent line to $\vec{r}\ \ t\ = \langle \sin 2t, t^2 - t, \cos 3t \rangle$ at the point where it intersects the *x* axis.
5. If $\vec{r}\ \ t\ \neq \vec{0}$, show that $\frac{d}{dt}\left|\vec{r}\ \ t\ \right| = \frac{\vec{r}\ \ t\ \bullet \vec{r}\,'\ \ t}{\left|\vec{r}\ \ t\ \right|}$
6. If $\vec{u}\ \ t\ = \vec{r}\ \ t\ \cdot \left[\vec{r}\,'\ \ t\ \times \vec{r}\,''\ \ t\ \right]$, show that $\vec{u}\,'\ \ t\ = \vec{r}\ \ t\ \bullet \left[\vec{r}\,'\ \ t\ \times \vec{r}\,'''\ \ t\ \right]$
7. Find the length of the curve $\vec{r}\ \ t\ = \langle 3\cos\ \ 3t\ \ , -t, 3\sin\ \ 3t\ \ \rangle$ on the interval $0 \le t \le \frac{2\pi}{3}$.
8. Suppose $\vec{r}\ \ t\ = \langle t^3, \sin t - t\cos t, \cos t + t\sin t \rangle$ where $t > 0$.
   a. Find the unit tangent and unit normal vectors $\hat{T}\ \ t$ and $\hat{N}\ \ t$.
   b. Find the curvature.
9. Find the curvature of $y = \sec\ \ x$.
10. Find the curvature of $\vec{r}\ \ t\ = \langle 1 + t^3, t + t^2 \rangle$.
11. Find equations for the normal plane and the osculating plane of $\vec{r}\ \ t\ = \langle 3\sin\ \ 2t\ \ , t, 3\cos\ \ 2t\ \ \rangle$ at $\ 0, 0,\ \left(0, \frac{\pi}{2}, -3\right)$.
12. Show that the circular helix $\vec{r}\ \ t\ = \langle 3\sin\ \ 2t\ \ , t, 3\cos\ \ 2t\ \ \rangle$ has constant curvature and constant torsion. The torsion is $\tau = \frac{\vec{r}\,' \times \vec{r}\,''\ \bullet \vec{r}\,'''}{\left|\vec{r}\,' \times \vec{r}\,''\right|^2}$.
13. Find the velocity, acceleration and speed of a particle with the given position function $\vec{r}\ \ t\ = \langle 1 + 2\cos 3t, 4\sin 3t, t \rangle$. Sketch the path of the particle a draw the velocity and acceleration vectors for the specified value of *t*.
14. Show that if a particle moves with a constant speed then the velocity and acceleration vectors are perpendicular.
15. A force with magnitude 10(1+sin(*t*)) Newtons pushes downward on a 4 kg mass. The mass starts at (0,0,100) and has an initial velocity $< 1, -1, 1 >$. Where does its path intersect the *xy* plance?
16. Find equations for the osculating and normal plane of the vector valued function $\vec{r}\ \ t\ = \langle \sin\ \ \pi t\ \ , 3\cos\ \ \pi t\ \ , \sin\ \ 2\pi t\ \ \rangle$ where $t = ½$.
17. The law of conservation of angular momentum says that if angular momentum is constant then the torsion is zero. $\vec{L}\ \ t$ The angular momentum of a mass *m* with position vector $\vec{r}\ \ t$ is $\vec{L}\ \ t\ = m\vec{r}\ \ t\ \times \vec{r}\,'\ \ t$ and its torque is $\vec{\tau}\ \ t\ = m\vec{r}\ \ t\ \times \vec{r}\,''\ \ t$. Show that $\vec{L}\,'\ \ t\ = \vec{\tau}\ \ t$.
18. Describe the surface parameterized by $\vec{r}\ \ u, v\ = \langle u\sin 2u\cos 3v, u\cos 2u\cos 3v, u\sin 3v \rangle$
19. Parameterize the part of the hyperbolic paraboloid $z = x^2 - y^2$ that lies above the *xy* plane.
20. Find parametric equations of the toroid obtained by rotating about the *x*-axis the circle in the *yz* plane with center (0,0,*b*) and radius $a < b$.

$r(t) = \left\langle \frac{\cos t}{1+\sin^2 t}, \frac{\sin t \cos t}{1+\sin^2 t}, \cos 4t \right\rangle$

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1. Find a parameterization for the parabola formed by the intersection of the plane $z = x + 3y$ with the cone $z = 2\sqrt{x^2+y^2}$.

   ANS: If $z = z$ then $x+3y = 2\sqrt{x^2+y^2}$. Equating squares, we have that if $x+3y \geq 0$ then $x^2+6xy+9y^2 = 4x^2+4y^2 \Leftrightarrow y^2 + \frac{6}{5}xy = \frac{3}{5}x^2 \Leftrightarrow y^2 + \frac{6}{5}xy + \frac{9}{25}x^2 = \frac{24}{25}x^2$ $\Leftrightarrow y + \frac{3}{5}x = \pm\frac{2\sqrt{6}}{5}x \Leftrightarrow y = \left(\frac{-3\pm 2\sqrt{6}}{5}\right)x \Rightarrow z = \left(\frac{-4\pm 6\sqrt{6}}{5}\right)x.$. This means we have a degenerate parabola: one that is also a degenerate hyperbola: the graph is two lines through the origin: $\vec{r}(x) = \left\langle x, \left(\frac{-3+2\sqrt{6}}{5}\right)x, \left(\frac{-4-6\sqrt{6}}{5}\right)x. \right\rangle$ or $\vec{r}(x) = \left\langle x, \left(\frac{-3-2\sqrt{6}}{5}\right)x, \left(\frac{-4+6\sqrt{6}}{5}\right)x. \right\rangle$

2. Find a parameterization of the line segment from $\vec{r}(1)$ to $\vec{r}(2)$ where $r(t) = \langle t, t^2, t^4 \rangle$.

   ANS: $\vec{p}(t) = \langle 1+t, 1+3t, 1+15t \rangle$

3. Parameterize the curve at the intersection of the ellipsoid $x^2+9y^2+4z^2 = 1$ with the elliptical paraboloid $x = 9y^2+4z^2$.

   ANS: Substituting, $x^2 + x = 1 \Leftrightarrow x = \frac{-1\pm\sqrt{5}}{2}$. But clearly x is positive, so $x = \frac{-1+\sqrt{5}}{2}$ is a plane parallel to the yz coordinate plane that cuts the elliptical paraboloid in an ellipse: $9y^2+4z^2 = \frac{-1+\sqrt{5}}{2} \Leftrightarrow \frac{y^2}{\frac{-1+\sqrt{5}}{18}} + \frac{z^2}{\frac{-1+\sqrt{5}}{8}}$ which we can parameterize with

   $$r(t) = \left\langle \frac{-1+\sqrt{5}}{2}, \sqrt{\frac{-1+\sqrt{5}}{18}}\cos t, \sqrt{\frac{-1+\sqrt{5}}{8}}\sin t \right\rangle$$

4. Find parametric equations for the tangent line to $\vec{r}(t) = \langle \sin 2t, t^2 - t, \cos 3t \rangle$ at the point where it intersects the z axis.

   ANS: If $x = y = 0$ then $t = 0$. $\vec{p}(t) = \vec{r}(0) + t\cdot\vec{r}\,'(0) = \langle 0,0,1\rangle + t\langle 2,-1,0\rangle = \langle 2t, -t, 1\rangle$

5. If $\vec{r}(t) \neq \vec{0}$, show that $\frac{d}{dt}|\vec{r}(t)| = \frac{\vec{r}(t)\cdot\vec{r}\,'(t)}{|\vec{r}(t)|}$.

   ANS: Let $\vec{r}(t) = \langle x(t), y(t), z(t)\rangle$. Then

   $$\frac{d}{dt}|\vec{r}(t)| = \frac{d}{dt}\left(x^2+y^2+z^2\right)^{1/2} = \frac{1}{2}\left(x^2+y^2+z^2\right)^{-1/2}\left(2xx'+2yy'+2zz'\right) = \frac{xx'+yy'+zz'}{\left(x^2+y^2+z^2\right)^{1/2}}$$

6. If $\vec{u}(t) = \vec{r}(t)\cdot\left[\vec{r}\,'(t)\times\vec{r}\,''(t)\right]$, show that $\vec{u}\,'(t) = \vec{r}(t)\cdot\left[\vec{r}\,'(t)\times\vec{r}\,'''(t)\right]$

   ANS: $$\frac{d}{dt}\vec{u}(t) = \frac{d}{dt}\left(\vec{r}(t)\cdot\left[\vec{r}\,'(t)\times\vec{r}\,''(t)\right]\right) = \vec{r}(t)\cdot\frac{d}{dt}\left[\vec{r}\,'(t)\times\vec{r}\,''(t)\right] + \vec{r}\,'(t)\cdot\left[\vec{r}\,'(t)\times\vec{r}\,''(t)\right]$$
   $$= \vec{r}(t)\cdot\left[\vec{r}\,''(t)\times\vec{r}\,''(t) + \vec{r}\,'(t)\times\vec{r}\,'''(t)\right] + \vec{0} = \vec{r}(t)\cdot\left[\vec{0} + \vec{r}\,'(t)\times\vec{r}\,'''(t)\right]$$

7. Find the length of the curve $\vec{r}(t) = \langle 3\cos(3t), -t, 3\sin(3t)\rangle$ on the interval $0 \le t \le \frac{2\pi}{3}$.

ANS: $\int_0^{2\pi/3} |\vec{r}\,'(t)|\,dt = \int_0^{2\pi/3} \sqrt{81\sin^2(3t) + 1 + 81\cos^2(3t)}\,dt = \int_0^{2\pi/3} \sqrt{82}\,dt = \frac{2\pi\sqrt{82}}{3}$

8. Suppose $\vec{r}(t) = \langle t^3, \sin t - t\cos t, \cos t + t\sin t\rangle$ where $t > 0$.

a. Find the unit tangent and unit normal vectors $\hat{T}(t)$ and $\hat{N}(t)$.

ANS: $\vec{r}\,'(t) = \langle 3t^2, t\sin t, t\cos t\rangle \Rightarrow |\vec{r}\,'(t)| = \sqrt{9t^4 + t^2} \Rightarrow \hat{T}(t) = \left\langle \frac{3t}{\sqrt{9t^2+1}}, \frac{\sin t}{\sqrt{9t^2+1}}, \frac{\cos t}{\sqrt{9t^2+1}} \right\rangle$

$$\hat{T}\,'(t) = \left\langle \frac{3\sqrt{9t^2+1} - \frac{27t^2}{\sqrt{9t^2+1}}}{9t^2+1}, \frac{\cos t\sqrt{9t^2+1} - \sin t\frac{9t^2}{\sqrt{9t^2+1}}}{9t^2+1}, \frac{-\sin t\sqrt{9t^2+1} - \cos t\frac{9t^2}{\sqrt{9t^2+1}}}{9t^2+1} \right\rangle$$

$$= \left\langle \frac{3}{(9t^2+1)^{3/2}}, \frac{(9t^2+1)\cos t - 9t^2\sin t}{(9t^2+1)^{3/2}}, \frac{-(9t^2+1)\sin t - 9t^2\cos t}{(9t^2+1)^{3/2}} \right\rangle$$

whence

$$|\hat{T}\,'(t)| = \sqrt{\frac{9 + \left((9t^2+1)\cos t - 9t^2\sin t\right)^2 + \left(-(9t^2+1)\sin t - 9t^2\cos t\right)^2}{(9t^2+1)^3}} = \frac{\sqrt{9 + (9t^2+1)^2 + (9t^2)^2}}{(9t^2+1)^{3/2}}$$

Thus $\hat{N}(t) = \frac{\hat{T}\,'(t)}{|\hat{T}\,'(t)|} = \frac{\langle 3, (9t^2+1)\cos t - 9t^2\sin t, -(9t^2+1)\sin t - 9t^2\cos t\rangle}{\sqrt{9 + (9t^2+1)^2 + (9t^2)^2}}$

b. Find the curvature.

ANS: $\kappa = \frac{|r' \times r''|}{|r'|^3} = \frac{|\langle -t^2, 3t^2\cos t + 3t^3\sin t, -3t^2\sin t + 3t^3\cos t\rangle|}{(9t^4+t^2)^{3/2}} = \frac{\sqrt{10 + 9t^2}}{t(9t^2+1)^{3/2}}$

9. Find the curvature of $y = \sec(x)$.

ANS: $\kappa = \frac{|f''(x)|}{\sqrt{1 + (f'(x))^2}} = \frac{|\sec x|(\tan^2 x + \sec^2 x)}{\sqrt{1 + \sec^2 x\tan^2 x}} = \frac{|\sec x|(1 + 2\tan^2 x)}{\sqrt{1 + \tan^2 x + \tan^4 x}}$

10. Find the curvature of $\vec{r}(t) = \langle 1 + t^3, t + t^2\rangle$.

ANS: To use the formula, we need to embed the function in 3-space: $\vec{r}(t) = \langle 1 + t^3, t + t^2, 0\rangle$ so that

$$\kappa = \frac{|r' \times r''|}{|r'|^3} = \frac{|\langle 3t^2, 1+2t, 0\rangle \times \langle 6t, 2, 0\rangle|}{(9t^4 + (1+2t)^2)^{3/2}} = \frac{|\langle 0, 0, 6t^2 - 6t - 12t^2\rangle|}{(9t^4 + 4t^2 + 4t + 1)^{3/2}} = \frac{|6t(t+1)|}{(9t^4 + 4t^2 + 4t + 1)^{3/2}}$$

11. Find equations for the normal plane and the osculating plane of $\vec{r}(t) = \langle 3\sin(2t), t, 3\cos(2t)\rangle$ at $\left(0, \frac{\pi}{2}, -3\right)$.

ANS: $\vec{r}\,'\left(\frac{\pi}{2}\right) = \langle 6\cos(\pi), 1, -6\sin(\pi)\rangle = \langle -6, 1, 0\rangle$ and

$\vec{r}\,''\left(\frac{\pi}{2}\right) = \langle -12\sin(\pi), 0, -12\cos(\pi)\rangle = \langle 0,0,12\rangle$ so a vector normal to the osculating plane is $\vec{n} = \langle -6,1,0\rangle \times \langle 0,0,12\rangle = 12\langle 1,6,0\rangle$. Thus an equation for the plane is $x + 6y = 3\pi$

12. Show that the circular helix $\vec{r}(t) = \langle 3\sin(2t), t, 3\cos(2t)\rangle$ has constant curvature and constant torsion. The torsion is $\tau = \dfrac{(\vec{r}\,'\times\vec{r}\,'')\cdot\vec{r}\,'''}{\left|\vec{r}\,'\times\vec{r}\,''\right|^2}$.

    ANS:
13. Find the velocity, acceleration and speed of a particle with the given position function $\vec{r}(t) = \langle 1+2\cos 3t, 4\sin 3t, t\rangle$. Sketch the path of the particle a draw the velocity and acceleration vectors for the specified value of t.
14. Show that if a particle moves with a constant speed then the velocity and acceleration vectors are perpendicular.
15. A force with magnitude 10(1+sin(t)) Newtons pushes downward on a 4 kg mass. The mass starts at (0,0,100) and has an initial velocity $< 1, -1, 1 >$. Where does its path intersect the xy plance?
16. Find equations for the osculating and normal plane of the vector valued function $\vec{r}(t) = \langle \sin(\pi t), 3\cos(\pi t), \sin(2\pi t)\rangle$ where $t = ½$.
17. The law of conservation of angular momentum says that if angular momentum is constant then the torsion is zero. $\vec{L}(t)$ The angular momentum of a mass m with position vector $\vec{r}(t)$ is $\vec{L}(t) = m\vec{r}(t)\times\vec{r}\,'(t)$ and its torque is $\vec{\tau}(t) = m\vec{r}(t)\times\vec{r}\,''(t)$. Show that $\vec{L}\,'(t) = \vec{\tau}(t)$.
18. Describe the surface parameterized by $\vec{r}(u,v) = \langle u\sin 2u\cos 3v, u\cos 2u\cos 3v, u\sin 3v\rangle$
19. Parameterize the part of the hyperbolic paraboloid $z = x^2 - y^2$ that lies above the xy plane.
20. Find parametric equations of the toroid obtained by rotating about the x-axis the circle in the yz plane with center (0,0,b) and radius $a < b$.